UDC 629.764

AN APPLICATION OF DIFFERENTIAL TRANSFORMATION FOR OPTIMAL CONTROL OF NONLINEAR PROCESSES

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Abstract. *In paper, the differential transformation is applied for solving problems of nonlinear optimal control processes by dynamic objects. An approach based on the multi-step differential transform method and the Adomian polynomials for approximation of nonlinear terms of differential equations that describe dynamic process is proposed. It is offered an optimization model of multi-step control process. The model advantage is the possibility of process simulation of optimal control with piecewise continuous functions, determination of an optimal control and state trajectories without using numerical integration methods of differential equations of object dynamics. Herewith, the analytical solution of a problem is allowed, which essentially reducing amount of calculation.*

Keywords*: optimization, differential transforms, Adomian polynomials, modified method, multi-step control.*

Introduction

For solving applied control problems in various fields of science and engineering are used the mathematical models of dynamic processes optimization. The problem of control optimization of nonlinear processes is one of the most complex problems in the theory of optimal control processes. For construction of corresponding mathematical models frequently apply nonlinear differential equations, which, in general case, don't have analytical solution and are solved by various numerical and numerical-analytical methods [1-4].

1. Analysis of the research and publications

An application of the majority well known methods for solving nonlinear differential equations and corresponding nonlinear boundary value problems is associated with overcoming several mathematical and computational difficulties. One way to overcome given difficulties is application of the differential transform method (DTM) [5-7]. It can be applied directly to solve of nonlinear differential equations without preliminary linearization, eliminates dependence of variables from time argument and admits the possibility to obtain analytical solution. The mathematical apparatus of differential transforms applies for solving nonlinear boundary value problems [8-10], linear and non-linear problems of optimal control [11-15].

The area of DTM application is limited by a class of analytic functions and small length of solution interval. At the same time, the optimal control problems require, in the general case, arbitrary piecewise continuous controls on the big intervals. For extending the search interval of solution and propagation of the area of DTM application to class of piecewise continuous functions applies the conception of multi-step DTM (MsDTM) [16-19]. The given conception consists in dividing of entire interval into sub-intervals, searching over each sub-interval the solution by DTM in a class of analytical functions and obtaining the general solution of equation as sum of solutions over sub-intervals in a class of piecewise continuous functions.

Frequently at solving nonlinear differential equations occur mathematical difficulties associated with the complex nonlinearity of equations. These difficulties can be overcoming by using of Adomian polynomials. The basis of the given approach forms the decomposition of nonlinear differential equation into linear and nonlinear parts and approximation of complex nonlinearity part of equation by Adomian polynomials. This is considerably simplifies solving of nonlinear problems and extends the area of DTM application [20,21].

2. Research tasks

In paper, we consider questions of optimization of nonlinear control processes based on the MsDTM conception with using Adomian polynomials that allows to extend the area of DTM application to optimal control processes of nonlinear systems described by piecewise analytical functions.

3. Optimal control problem

Let’s the equation of motion of an object is described by vector differential equation:

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)),\ t \in [t_0, T], \qquad (1)$$

where $x(t) \in R^n$ is n-measurement of state vector, $u(t) = R^m$ is m-measurement of control vector, $m \prec n$, $f \in R^n$ is continuous and continuously differentiable on plurality variables $x_1,...,x_n$ the vector function of generalized force. The final value of time T can be given or non-fixed. In terminal control problems, the vector of final state $x(T)$ should be satisfied the boundary condition in the form of l-measurement vector equation $(l \le n)$:

$$S[T, x(T)] = 0 . \qquad (2)$$

In some cases, on the state and control vectors can be imposed the restriction in the form of r-measurement vector inequation:

$$G[t, x(t), u(t)] \le 0 . \qquad (3)$$

The quality of control process is estimated by the functional:

$$J = Q[T, x(T)] + \int_{t_0}^{T} E(t,x,u)dt , \qquad (4)$$

where the given functions E, Q have continuous partial derivatives with respect to its own parameters.

The optimal control problem is defined as follows. For given equations of motion of an object (1), boundary conditions (2) and restriction (3) it's required to find control u, that transfers the object from the initial state $x(t_0)$ to final (terminal) state $x(T)$ and provides optimization of the functional (4).

Given problem can be reduced to solving system of differential equations in two ways: by either Pontryagin maximum principle (PMP) and dynamic programming (DP) or their modifications.

Pontryagin maximum principle

The PMP states the necessary conditions of optimality [22].

According to the maximum principle, we introduce the Hamiltonian $H(t,x,u,p)$ function, which depended from the vector of adjoint variables $p = p(t)$:

$$H(t,x,u,p) = E(t,x,u) + p^T \cdot f(t,x,u) . \qquad (5)$$

Optimal trajectory is defined from solving of the system of differential equations:

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \frac{dp}{dt} = -\frac{\partial H}{\partial x}, \qquad (6)$$

with the boundary conditions:

$$x(t_0) = x_0, \; p(T) = -\frac{\partial Q}{\partial x} . \qquad (7)$$

The control optimizing the functional (4) is realized by control vector $u(t)$, defined from condition:

$$\frac{\partial H}{\partial u} = 0. \qquad (8)$$

The application of given approach to optimization of nonlinear control processes reduces to a two-point boundary value problem, which in many cases doesn't have an analytical solution and is solved by various approximate methods.

Dynamic programming

We choose and fix the arbitrary instant $t \in [t_0, T]$ and consider the auxiliary control problem (1-4) over the interval $[t, T]$. Denote as $V(t,x)$ the minimum value of functional (4) at the initial condition $x(t) = x$. Then for considered optimal control problem, the function $V(t,x)$ satisfies the Bellman equation [23]:

$$\min_u \left\{ \frac{\partial V(t,x)}{\partial t} + f(t,x,u)\frac{\partial V(t,x)}{\partial x} + \atop + E(t,x,u) \right\} = 0 . \qquad (8)$$

with the boundary condition:

$$V(T, x(T)) = Q(x(T)) . \qquad (9)$$

Thus, the application of DP leads to necessity of solving a complex nonlinear differential equation in partial derivates (8), therewith with a minimum sign under conditions (1), (2) and (9), which in many cases it is difficult to solve. Herewith, an essential drawback of the method is the assumption of differentiability of an unknown function $V(t,x)$, that can't be verified by the state equations.

4. Differential transform method

The DTM allows to replace in the mathematical model of physical process the functions $x(t)$ continuous argument t by their spectral models in the form of discrete functions $X(k)$ of integer argument $k = 0,1,2,...$.

The differential transformation of function $x(t)$ are defined as:

$$\underline{x(t)} = X(k) = \frac{H^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} , \qquad (10)$$

where $x(t)$ is the original function; $X(k)$ is the differential image of original (differential spectrum);

H is the scale stationary value having dimensionality of argument t and usually chosen equal to the interval $0 \le t \le H$, on which the function $x(t)$ is considered; the line below is the character of DT transformations.

The inverse transformation allows to obtain the original $x(t)$ by the image $X(k)$ in the form of Taylor series:

$$x(t) = \sum_{k=0}^{\infty} X(k) t^k. \qquad (11)$$

Generally, in actual application of differential transformation the function $x(t)$ is defined as finite series:

$$x(t) \approx y(t) = \sum_{k=0}^{N} X(k) t^k. \qquad (12)$$

5. The multi-step DTM

Consider the nonlinear ordinary differential equation of order m:

$$f(t, x, x', ..., x^{(m)}) = 0, \ t \in [t_0, T] \qquad (13)$$

subject to the given initial conditions:

$$x^{(p)}(t_0) = c_p, \ p = 0, 1, ..., m-1. \qquad (14)$$

Let's divide the total time interval $[t_0, T]$ into r given sub-intervals of length $T_q = t_q - t_{q-1}, \ q = \overline{1, r}$, $\sum_{q=1}^{r} T_q = T - t_0$. Sub-intervals have the equal step-size $H = T / r$ and $t_q = qH$. Applying the MDT to the problem (13) - (14) over the first sub-interval $[t_0, t_1]$ we will obtain the solution in the form:

$$x_1(t) = \sum_{k=0}^{P} X_1(k) t^k, \ t \in [t_0, t_1].$$

Taking into account the initial condition $x_1^{(p)}(t_0) = c_p$ and the expression (10) we can find for the first sub-interval all values $X_1(k)$, $k = 0, 1, 2, ..p$. For $q \ge 2$ and at each following sub-interval $[t_{q-1}, t_q]$ we will use the initial condition $x_q^{(p)}(t_{q-1}) = x_{q-1}^{(p)}(t_{q-1})$. Then the expression (10) for the q^{th} sub-interval will be following:

$$X_q(p) = \frac{H^p}{p!} \left[\frac{d^p x_{q-1}(t)}{dt^r} \right]_{t = t_{q-1}}, \ p \ge 0.$$

Now applying the DTM to the problem (13) - (14) over the interval $[t_{q-1}, t_q]$. The process is repeated and, in results, we obtain the sequence of approximate solutions $x_q(t), \ q = 1, ..., r$ for the solution $x(t)$, where

$$x_q(t) = \sum_{k=0}^{P} X_q(k) (t - t_{q-1})^k, \ t \in [t_{q-1}, t_q].$$

Here $N = P \cdot r$.

Finally, at using of the MsDTM we obtain the following solution:

$$x(t) = \begin{cases} x_1(t) \approx y_1(t), \ t \in [t_0, t_1] \\ x_2(t) \approx y_2(t), \ t \in [t_1, t_2] \\ ..... \\ x_r(t) \approx y_r(t), t \in [t_{r-1}, t_r] \end{cases}. \qquad (15)$$

If $r = 1$ then $H = T$ and the MsDTM reduces to the traditional DTM.

The estimate of accuracy of approximate solution by the MsDTM

The quantity of accounted discretes for restoring of the solution as Taylor series is one of the most essential factors that effect on the solution accuracy obtained. Restriction of given quantity of discretes leads to error of results obtained.

Consider the nonlinear boundary value problem (13) with initial condition (14). The solution of given problem will consider over the interval $t_0 \le t \le T$, where the length of interval $L = T - t_0$ is selected inside the radius R of convergence of Taylor series, i.e. $0 \le L \prec R$. Assume that analytic function $x(t)$ is continuously differentiated in any point $t \in [t_0, T]$, has derivatives of m^{th} order, which are limited in total for any whole $m \ge 1$ so that,

$$\left| x^{(m)}(t) \right| \le C < +\infty, \ \ t \in [t_0, T]. \qquad (16)$$

The upper bound of error estimate $|\varepsilon_0| = |x(t)\text{-} y(t)|$ of the DTM (1) is given by the expression [24]:

$$|\varepsilon_0| \le \frac{L^{s+1}}{(s+1)!} \sup_{0 < t_l < L} \left| x^{(s+1)}(t_l) \right|, \qquad (17)$$

where s is the quantity of accounted discretes in the solution.

Taking into account the constraint (16), the expression (17) can be written as:

$$|\varepsilon_0| \le C \frac{L^{s+1}}{(s+1)!} = |\hat{\varepsilon}_0|. \qquad (18)$$

For the MsDTM, the expression (17) for the upper bound of error estimate $\left|\varepsilon_q\right| = \left|x_q(t) - y_q(t)\right|$ over q th sub-interval with taking into account s discretes can be written as:

$$\left|\varepsilon_q\right| \le \frac{(L/r)^{s+1}}{(s+1)!} \sup_{\xi \in [t_q, t_{q-1}]} \left|x_q^{(s+1)}(\xi)\right|, \ q = 1, \ldots, r. \quad (19)$$

On the ground of the constraint (16), can make the conclusion, that over q th sub-interval

$$C_q = \sup_{\xi \in [t_q, t_{q-1}]} \left|x_q^{(s+1)}(\xi)\right| \le C < +\infty. \quad (20)$$

Really, if the $(s+1)$ th derivative of function $x_q(t)$ achieves the maximum value over the interval $[t_0, t_q]$, then $C_q = C$, otherwise $C_q < C$.

The error estimate (19) with taking into account the constraint (20) can be written as

$$\left|\varepsilon_q\right| \le C \frac{(L/r)^{s+1}}{(s+1)!}. \quad (21)$$

From the expression (21) follows that obtained error at dividing the entire interval into equal r sub-intervals, is the same over sub-intervals and depends only from the quantity of accounted discretes s. Convert the given estimate to the relative error estimate. Let us select as a comparison base the error (18) for the DTM. Then for the relative error on q th sub-interval obtain:

$$\left|\frac{\varepsilon_q}{\hat{\varepsilon}_0}\right| \le \frac{(L/r)^{s+1}}{L^{s+1}}, \quad q = 1, \ldots, r. \quad (22)$$

Consider the full relative error of the MsDTM $\varepsilon_s = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_r$ over the entire interval in relation to the error ε_0 of the DTM:

$$\left|\frac{\varepsilon_s}{\hat{\varepsilon}_0}\right| = \left|\frac{\varepsilon_1}{\hat{\varepsilon}_0}\right| + \left|\frac{\varepsilon_2}{\hat{\varepsilon}_0}\right| + \ldots + \left|\frac{\varepsilon_r}{\hat{\varepsilon}_0}\right|. \quad (23)$$

From expression (22) follows, that components of relative errors are changed in the bounds:

$$0 \le \left|\frac{\varepsilon_{s_i}}{\hat{\varepsilon}_0}\right| \le 1, \quad i = 1, 2, \ldots, r.$$

Bigger deviation of approximate solution from exact solution, usually, falls on the end of time interval. At that, it will be maximum in the case of the same signs ε_{s_i}. Then, taking into account (22), the expression (23) will be following:

$$\left|\frac{\varepsilon_s}{\hat{\varepsilon}_0}\right| \le \left|\frac{1}{r^{s+1}}\right| + \left|\frac{1}{r^{s+1}}\right| + \ldots + \left|\frac{1}{r^{s+1}}\right| = \frac{1}{r^s} = r^{-s}.$$

This means that the upper bound of error estimate of the MsDTM in r^s time less than the upper bound of error estimate of the DTM at dividing of given interval into r sub-intervals of equal step-size, i.e.:

$$\left|\varepsilon_s\right| \le r^{-s} \left|\hat{\varepsilon}_0\right|,$$

where s is the quantity of accounted discretes of differential spectrum $X(k)$ above the zeroth discrete $X(0)$, i.e. the quantity s is equal the number of the last accounted discretes of differential spectrum $X(k)$. The analysis of obtained expression shown, that with increasing of quantity of accounted discretes s, the upper bound of summary error is reduced on the exponential rule and at $s \to \infty$ reduced to the zeroth lower bound. Therefore, the range of changing of summary error at dynamical processes simulation using the MsDTM at dividing the interval into r sub-intervals of equal length is defined by constraints:

$$0 \le \left|\varepsilon_s\right| \le r^{-s} \cdot \left|\hat{\varepsilon}_0\right|, \quad (24)$$

where $\left|\hat{\varepsilon}_0\right|$ is defined by expression (18).

From the expression (24) follows, that the MsDTM at the restricted quantity of discretes s of differential spectrum $X(k)$ gives the possibility to get more exact solution of boundary value problem (13) – (14) at the point $t = T$ at condition execution (16), than the DTM.

6. The modified multi-step DTM

At the heart of the modified MsDTM (MMsDTM) lies the combination of MsDTM and Adomian polynomials.

Consider the nonlinear ordinary differential first-order equation:

$$\dot{x}(t) = g[t, x(t)] + f[t, x(t)], t \in [0, T] \quad (25)$$

with given initial condition $x(0)$, where $g[t, x(t)]$, $f[t, x(t)]$ are respectively a linear and nonlinear parts of equation with respect to x.

Let us divide the given interval $[0, T]$ into r sub-intervals. In accordance with Adomian polynomials and taking into account the features of DT-transformation, components of differential image $F_q(k)$ of nonlinear function of desired differential equation for q^{th} sub-interval ($q=\overline{1,r}$) are defined in the form [16]:

$$
\begin{aligned}
&F_q(0)= f(x_q(0))= f(X_q(0))= f(x_{q0}),\\
&F_q(1)=\frac{d}{dt}f(x_q(t))\Big|_{t=0}= x'_q(0)f^{(1)}(x_q(0))=\\
&= X_q(1)f^{(1)}(X_q(0)),\\
&F_q(2)= X_q(2)f^{(1)}(X_q(0))+\\
&+\frac{1}{2!}(X_q(1))^2 f^{(2)}(X_q(0)),\\
&F_q(3)= X_q(3)f^{(1)}(X_q(0))+ X_q(1)X_q(2)f^{(2)}(X_q(0))+\\
&+\frac{1}{3!}(X_q(1))^3 f^{(3)}(X_q(0)),\\
&F_q(4)= X_q(4)f^{(1)}(X_q(0))+(X_q(1)X_q(3)+\\
&+\frac{1}{2!}(X_q(2))^2)f^{(2)}(X_q(0))+\\
&+\frac{1}{2!}(X_q(1))^2 X_q(2)f^{(3)}(X_q(0))+\\
&+\frac{1}{4!}(X_q(1))^4 f^{(4)}(X_q(0)),\\
&F_q(5)= X_q(5)f^{(1)}(X_q(0))+(X_q(2)X_q(3)+\\
&+X_q(1)X_q(4))f^{(2)}(X_q(0))+\frac{1}{2!}(X_q(1))^2 X_q(3)+\\
&+X_q(1)(X_q(2))^2)f^{(3)}(X_q(0))+\\
&+\frac{1}{3!}(X_q(1))^3 X_q(2)f^{(4)}(X_q(0))+\\
&+\frac{1}{5!}(X_q(1))^5 f^{(5)}(X_q(0)),\ldots
\end{aligned}
\qquad (26)
$$

In [25] shown, that such presentation of the components of differential image of nonlinear function can be applied for any types of nonlinearity of differential equations.

Taking into account the existence of effective methods of Adomian polynomials calculation, the given approach allows to overcome the mathematical difficulties at calculation of differential images of complex nonlinearities and essentially reduces the computational cost at finding the approximate solution of boundary problems which are described by nonlinear differential equations.

7. Example of nonlinear differential equation solution

Consider the nonlinear ordinary differential equation with the quadratic source term:

$$\frac{dx(t)}{dt}=2x(t)-x^2(t)+1,\quad x(0)=0. \qquad (27)$$

The exact solution of given equation is given by [16]: $x(t)=1+\sqrt{2}\tanh\left(\sqrt{2}t+\frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$.

Taking into account the features of DTM, we write the equation (27) in the spectral form:

$$
\begin{aligned}
&(k+1)X(k+1)=2X(k)-F(k)+\sigma(k),\\
&X(0)=0,
\end{aligned}
\qquad (28)
$$

where $F(k)$ is the differential image of nonlinear function $f(x)=x^2$, $\sigma(k)=\begin{cases}1, & k=0\\ 0, & k\ge 1.\end{cases}$

The solution of equation (27) we will consider over interval $t\in[0,2]$. Let us divide the given interval into r sub-intervals of equal step-size $H=2/r$ and write the equation (28) for each sub-interval:

$$
\begin{aligned}
&(k+1)X_q(k+1)=2X_q(k)-F_q(k)+\sigma(k),\\
&X_1(0)=x_1(0)=x(0)=0,\ x_q(t_{q-1})=\\
&=x_{q-1}(t_{q-1}),\ q=2,\ldots,r.
\end{aligned}
\qquad (29)
$$

Components of differential image of function $f(x)=x^2$ for q^{th} sub-interval are defined in the form:

$$
\begin{aligned}
&F_q(0)=X_q^2(0),\ F_q(1)=2X_q(0)X_q(1),\\
&F_q(2)=X_q^2(1)+2X_q(0)X_q(2),\\
&F_q(3)=2X_q(0)X_q(3)+2X_q(1)X_q(2),\\
&F_q(4)=2X_q(0)X_q(4)+2X_q(1)X_q(3)+\\
&+X_q^2(2),\\
&F_q(5)=2X_q(0)X_q(5)+\\
&+2\big(X_q(2)X_q(3)+X_q(1)X_q(4)\big),\ldots
\end{aligned}
\qquad (30)
$$

Substituting values $F_q(k)$ in (29) and taking into account (12), we find the approximate solution of equation (27) over each sub-interval. Summing given solutions obtain the general solution of equation (27) on the given interval.

Figure 1 show the comparison between exact solution of given equation, solution by the DTM $(r=1)$ and solution by the proposed MMsDTM at dividing the given interval into 2, 4 and 10 subintervals.

The results are obtained using first 6 discretes of differential image of function $x(t)$.

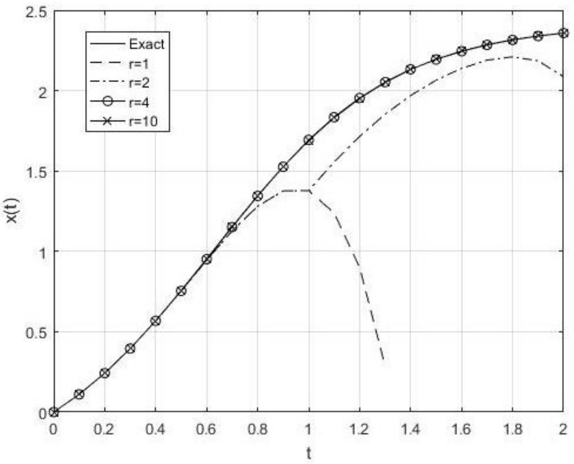

Fig. 1. Solution comparisons of equation (27)

The table 1 presents the relative error of solutions obtained.

8. The model of optimal multi-step dynamic control processes

In the literature, in considering the optimal nonlinear control processes, the main attention is given to questions either proof of solution existence or the necessary conditions for optimality and practically no considering is given to issues related to creation of effective methods of constructive optimization for practical applications. Herewith, as a rule, the programming solutions of optimal control problems are considered, that focused on discovering the potential capabilities of control systems and poorly adapted for actual control of dynamic object. The actual control implies the existence the feedback in control that allows effectively counteract the external perturbances [4].

Table 1

The relative error of solutions

t	Exact solution	The quantity of sub-intervals							
		p = 1		p = 2		p = 4		p = 10	
		Solution	ε_r	Solution	ε_r	Solution	ε_r	Solution	ε_r
0.0	0	0	0	0	0	0	0	0	0
0.1	0.110295196	0.110295177	8.06e-09	0.110295177	8.06e-09	0.110295177	8.06e-09	0.110295177	8.25e-09
0.2	0.241976799	0.241974042	1.17e-06	0.241974042	1.17e-06	0.241974042	1.17e-06	0.241974042	1.17e-06
0.3	0.395104849	0.395052593	2.22e-05	0.395052593	2.22e-05	0.395052592	2.22e-05	0.395101635	1.36e-06
0.4	0.567812166	0.567384160	1.82e-04	0.567384160	1.82e-04	0.567384159	1.82e-04	0.567802478	4.11e-06
0.5	0.756014393	0.753819406	9.31e-04	0.753819406	9.31e-04	0.753819406	9.31e-04	0.756004030	4.40e-06
0.6	0.953566217	0.945254325	3.53e-03	0.945254325	3.53e-03	0.951305884	9.59e-04	0.953557089	3.87e-06
0.7	1.152948967	1.127566235	1.08e-02	1.127566235	1.08e-02	1.150720089	9.45e-04	1.152940023	3.79e-06
0.8	1.346363655	1.280437785	2.80e-02	1.280437785	2.80e-02	1.344380575	8.41e-04	1.346365965	-9.80e-07
0.9	1.526911313	1.376068943	6.40e-02	1.376068943	6.40e-02	1.526090723	3.48e-04	1.526913446	-9.05e-07
1.0	1.689498392	1.377777000	1.32e-01	1.377777000	1.32e-01	1.693269326	-1.60e-03	1.689501594	-1.36e-06
1.1	1.831240782	1.238484565	-	1.555603647	1.17e-01	1.834477382	-1.37e-03	1.831243482	-1.15e-06
1.2	1.951360119	0.899095559	-	1.714827696	1.00e-01	1.954060293	-1.15e-03	1.951356182	1.67e-06
1.3	2.050735693	0.286759215	-	1.852929091	8.39e-02	2.052844842	-8.95e-04	2.050732449	1.38e-06
1.4	2.131326610	-0.686977929	-	1.969328780	6.87e-02	2.132338811	-4.29e-04	2.131321347	2.23e-06
1.5	2.195633294	-2.128132031	-	2.064533078	5.56e-02	2.193498948	9.05e-04	2.195629127	1.77e-06
1.6	2.246285959	-4.162353955	-	2.138696629	4.56e-02	2.244614083	7.09e-04	2.246283265	1.14e-06
1.7	2.285778286	-6.937113270	-	2.189603939	4.08e-02	2.284481384	5.50e-04	2.285776201	8.84e-07
1.8	2.316324737	-10.623994260	-	2.210069511	4.51e-02	2.315340893	4.17e-04	2.316324057	2.89e-07
1.9	2.339806374	-15.421103928	-	2.184756548	-	2.339163535	2.73e-04	2.339805856	2.20e-07
2.0	2.357771653	-21.555592000	-	2.086414259	-	2.357772701	-4.44e-07	2.357771729	-3.19e-08

An application of mathematical apparatus of differential transformations allows to construct the effective model of optimization of dynamic control processes [26]. Below, the model of optimal multi-step dynamic control processes based on the MMsDTM has constructed.

According the conception of multi-step control, whole control process on the interval $[t_0, T]$ we divide into r given time sub-intervals of length $T_q = t_q - t_{q-1}$, $\sum_{q=1}^{r} T_q = T - t_0$. Assume, that inside each time sub-interval, the variables of state vector are continuous and all discontinuities happen at boundaries of given time sub-intervals.

Assume, that at each time sub-interval the object motion is described by vector nonlinear differential equation:

$$\frac{dx_q}{dt} = g_q(t, x_q, u_q, \text{v}_q) + f_q(t, x_q), \quad x_q(t_{q-1}) = x_q^0,$$

$$t \in [t_{q-1}, t_q]; \; q = \overline{1, r}, \qquad (31)$$

where x_q is n-measurement state vector; u_q is m-measurement control vector; v_q is ℓ-measurement vector of turbulence; g_q is continuous and continuous differentiable on plurality variables t, x_q, u_q, v_q at each time sub-interval the vector function of linear generalized force; f_q is continuous and continuous differentiable on plurality variables t, x_q at each time sub-interval the vector function of nonlinear by parameter x_q of generalized force; t is current time inside q th sub-interval.

The problem of terminal control consists in the object translation from given initial state $x(t_0) = x^0$ to final (terminal) state $x_r(T)$, which is determined at the point of time $t = T$ z-measurement $(z \le n)$ vector equation:

$$S[x_r(T), T] = 0. \qquad (32)$$

The quality of control process is estimated by the functional:

$$I = Q[x_r(T), T] + \sum_{q=1}^{r} \int_{t_0}^{T} E_q(t, x_q, u_q, v_q) dt, \qquad (33)$$

where the given functions Q and E_q have continuous partial derivatives by x_q, u_q, v_q at each time interval. Assume, that restriction on state vectors and the control are taken into account during the selection of the functional type (33).

The conjugation of boundary and initial conditions of sub-intervals are set in the form of given boundary conditions:

$$\varphi_q\left[x_q(T_q), x^0_{q+1}; u_q(T_q), u^0_{q+1}; T_q\right] = 0, q = \overline{1, r}. \qquad (34)$$

The problem of multi-step terminal control (31)-(34) is solving in the following sequence. Primarily, the problem of optimal control $u_1(t)$ determination is solved over sub-interval $t_0 \le t \le t_1$ with initial condition of state vector $x_1(t_0) = x^0_1$. Solution of equation (31) at the point t_1 has the value $x_1(t_1)$. On the second stage, the problem of optimal control $u_2(t)$ determination is solved over the sub-interval $t_1 \le t \le t_2$ with initial condition $x_1(t_1) = x^0_2$ of the state vector. Solution of equation (31) at the point t_2 has the value $x_2(t_2)$. Constructed by such way the solution $x(t)$ and the control $u(t)$ are continuous at all points of sub-interval $t_0 \le t \le t_2$ and at the junction point t_1 of first and second sub-intervals. If we continuing this process over whole given interval $[t_0, T]$ obtain continuous and piecewise differentiable solution of the equation (31) and corresponded optimal control, which at given differential links (31), boundary conditions (32) and conditions of conjugation the boundary and initial conditions (34) are optimizing the functional (33) in absence of perturbance activity. Under actual conditions, the impact of an external environment $v_q(t)$ on the object motion dynamics can leads to essential terminal errors at the moment of control process termination upon program $u = u(t)$. With the purpose of neutralizing these disturbances on the next step is necessary to synthesize the algorithm of optimum by criterion (33) feedback control of the type $u = u(t, x)$, which at each time moment t uses information about current state $x(t)$ of dynamic object. The control with feedback provides the translation of dynamic object from an arbitrary initial state into terminal subjected to disturbances.

The optimal multi-step control synthesis with feedback can be executed by the method of closure of optimal program control $u = u(t)$ for an arbitrary current state of object. Primarily consider an undisturbed motion of the object. Let's select inside each sub-interval of control process a program control $u_q(\tau, A_q)$ from a class of analytic functions, where $A_q = (a_{q_1}, a_{q_2}, \dots a_{q_n})$ is the vector of free parameters, τ is the local time argument.

Differential transformations (1) of function $u_q(\tau, A_q)$ are determined at $H = T_q$ and $\tau = 0$ its differential spectrum as:

$$\underline{u_q}(\tau, A_q) = U_q(k, A_q) = \\ = \frac{T_q^k}{k!}\left[\frac{d^k u_q(t_{q-1} + \tau, A_q)}{dt^k}\right]_{\tau=0}. \qquad (35)$$

Vector differential equation of object dynamic (31) based on differential transformations (1) in the images field can be presented as the following spectral model:

$$X_q(k+1, A_q, X^0_q) = \\ = \frac{T_q}{k+1}\begin{Bmatrix} \underline{g_q}\left[T_q, X_q(k, A_q, X^0_q), U_q(k, A_q)\right] + \\ + \underline{f_q}[T_q, X_q(k, A_q, X^0_q)] \end{Bmatrix}, \qquad (36)$$

$$X_q(0) = X^0_q(A_{q-1}, A_{q-2}, \dots, A_1);$$

$$X_1(0) = X^0_1 = x_0; q = \overline{1, r}$$

Spectral model (36) has a universal character and can be applied for solving control problems of different dynamic objects. Given model is a recursion expression that gives the possibility from the differential spectrum (35) of function $u_q(\tau, A_q)$ to form the differential spectrum $X_q(k, A_q, X_q^0)$ of state vector $x_q(t)$. In the model (36) the components of differential image of nonlinear function $\underline{f_q}$ are approximated by Adomian polynomials.

Let's use the property of the differential transformations, according to which the algebraic total of all components (discretes) differential spectrum of any analytical function at the point $t = t_v$ is equal to zero discrete of a differential spectrum of function at the point $t_{v+1} = t_v + h$ or value of the original of function at the same point:

$$\sum_{k=0}^{\infty} X_v(k) = X_{v+1}(0) = x(t_v + h). \quad (37)$$

From the relation (37) at $t_v = t_{q-1}$ and $h = T_q$ we determine a state vector at the end of each q th time interval of control process:

$$x_q(T_q, A_q, x_q^0) = \sum_{k=0}^{\infty} X_q(k, A_q, X_q^0), q = \overline{1,r} \quad (38)$$

Then the equation of the final state (32) of whole control process with taking account of the expression for conjugation of boundary and initial intervals of control process (34) and also the expression for a state vector at the end of each q th time interval of control (38) is conversed as followed:

$$S[A_1, A_2, ..., A_r] = 0. \quad (39)$$

The given boundary condition in the implicit form defines z components of vector of free parameters A_q, $q = \overline{1,r}$ for q th time sub-interval and zr components for whole control process as functions from T_q and x_q^0.

The differential transformations (1) of functional (33) in view of differential spectra (35) and (36) allow presenting the given functional as the function of vector of free parameters A_q:

$$I(A_1, A_2, ..., A_r) = Q[A_1, A_2, ..., A_r] + \\ + \sum_{q=1}^{r} T_q \sum_{r=0}^{\infty} \frac{\underline{E}_q [T_q, X_q(k, A_q, X_q^0), U_q(k, A_q)]}{k+1} \quad (40)$$

The necessary requirements of an optimality of the function (40) enable to receive the system of equations for determining remaining $n - z$ components of vectors of free parameters for q th time sub-interval and $(n - z)r$ components for whole control process [27]:

$$\frac{\partial I(A_1, A_2, ..., A_r)}{\partial a_{qj}} = 0; \; q = \overline{1,r}; \; j = \overline{z+1,n} \quad (41)$$

The obtained system of the nonlinear equations (39) and (41) in the implicit form defines all components of a vector of free parameters of control $A = (A_1, A_2, ..., A_r)$ as function from a vector of an arbitrary initial state $x_0 = x_q(t_{q_0})$. As a result, for each sub-interval of control process in the implicit form, the nonlinear link of optimum program control $u_q[t, A(T_q, x_0)]$ with a vector of the initial state $x_o = x_q(t_{q_0})$, time t_{q_0} and time of sub-interval T_q of the multi-step control process. Under the impact of perturbations, the obtained control can't be applied over whole time sub-interval T_q, because it can be utilized for control only in the initial instant t_{q_0}. Thus, differential transformations allow to obtain for each sub-interval of control process in the analytic form the system of equations (39) and (41) for arbitrary values of the initial state $x_o = x_q(t_{q_0})$, instant t_{q_0} and time sub-interval T_q.

Under the impact of perturbations, the object continuously declines from the optimum program trajectory. In this case control $u_q[t, A(T_q, x_q)]$ for each sub-interval is calculated from the system of equations (39) and (41) for current values of time t_q and state $x_q(t_q)$. Thus, the continuous by the time solving the system of equations (39) and (41) allows to form for each sub-interval the closed-form law of terminal control in the form $u_q = u_q(t_q, x_q)$. Solving the system of equations (39) and (41) for each current instant t_q and the state $x_q(t_q)$ of dynamic object, located on the q th sub-interval subjected to turbulence, continuously sets control $u_q = u_q(t_q, x_q)$, linking current state $x_q(t_q)$ of dynamic object with boundary (terminal) conditions (32). In the closed circuit of control only the current value of control $u_q[t_q, A(T_q, x_q)]$ will be utilized which in the following instant is recalculated by the system of equations (39) and (41). It provides "flexible" adaptation of optimal trajectory of the object motion to the action of unknown turbulence factors $v_q(t)$.

If except the vector of optimum program control is necessary to define the components of state vector $x_q(\tau, A_q)$, that they can be obtained by the differential spectrum (39) as a truncated Taylor series or based on inverse polynomials of Legendre, Chebyshev, Fourier series and as combination of various approximating functions [5]. Free parameters of approximating functions are determined from comparison of differential spectra of state vector components with differential spectra of approximating functions.

It is necessary to note, that effectivity of model constructed is reduced with dimensionality increasing of vector of free parameters of control. Therefore, at solving specific problems it is worthwhile to limit ourselves to the low dimensionality of the given vector. It is also necessary to verify the existence of an extremum of the function (40) under sufficient optimality conditions.

9. Conclusions

The model of nonlinear optimal control processes by dynamic object based on the MsDTM with using Adomian polynomials for approximation of nonlinearities is constructed. The model advantage is an elimination of dependence of variables from time argument, the possibility to find optimal control and state trajectories without using the numerical methods of integration of differential equations of object dynamics. An analytical solution of control algorithm synthesis is allowed, and amount of calculation is reduced.

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ПРИМЕНЕНИЕ ДИФФЕРЕНЦИАЛЬНЫХ ПРЕОБРАЗОВАНИЙ К НЕЛИНЕЙНЫМ ОПТИМАЛЬНЫМ ПРОЦЕССАМ УПРАВЛЕНИЯ

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Аннотация. *Рассмотрено применение дифференциальных преобразований к решению задач нелинейных оптимальных процессов управления динамическими объектами. Предложен подход, основанный на многоэтапном методе дифференциальных преобразований с применением полиномов Адомиана для аппроксимации нелинейных членов дифференциальных уравнений, описывающих динамический процесс. Построена модель оптимизации многоэтапного процесса управления. Преимуществом модели является возможность моделирования процессов оптимального управления с кусочно-непрерывными функциями, находить оптимальное управление и фазовые траектории без использования численных методов интегрирования дифференциальных уравнений динамики объекта. При этом, допускается аналитическое решение проблемы, что значительно сокращает объем вычислений.*

Ключевые слова: *оптимизация, дифференциальные преобразования, полиномы Адомиана, модифицированный метод, многоэтапное управление.*

Received 28.07.2017

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